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Valid inequalities for sets defined by multilinear functions

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Abstract

We describe a class of valid inequalities for the bounded, nonconvex set described by all points within a $(n+1)$-dimensional hypercube whose $(n+1)$-st coordinate is equal to the product of the first n coordinates, for any $n \geq 2$. This set can be defined as $M_n = \{x \in R^{n+1} : x_{n+1} = x_1 x_2 \cdots x_n, l_i \leq x_i \leq u_i \forall i = 1, 2 \ldots, n+1\}$, with l_i and u_i constants. Approximating the convex hull of M_n through linear inequalities is essential to a class of exact solvers for nonconvex optimization problems, namely those which use Linear Programming relaxations to compute a lower bound on the problem. Together with the well-known McCormick inequalities, these inequalities are valid for the convex hull of M_2. There are infinitely many such inequalities, given that the convex hull of M_2 is not, in general, a polyhedron. The generalization to M_n for $n > 2$ is straightforward, and allows us to define strengthened relaxations for these higher dimensional sets as well.

Keywords: nonconvex optimization, multilinear functions, convex hull.

1. Multilinear sets

Consider the following nonconvex, bounded set

$$M_n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = \prod_{i=1}^{n} x_i, \ell_i \leq x_i \leq u_i, i = 1, 2 \ldots, n+1\},$$

where l and u are $(n+1)$-vectors. We assume $0 < \ell_i < u_i$ for all $i = 1, 2 \ldots, n+1$. The set M_n is nonconvex as the function $\xi_n(x) = \prod_{i=1}^{n} x_i$ is neither convex nor concave – its *epigraph* $\text{epi}(M) = \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq \prod_{i=1}^{n} x_i, 0 < \ell_i \leq x_i \leq u_i, i = 1, 2 \ldots, n\}$ and its *hypograph* $\text{hyp}(M) = \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq \prod_{i=1}^{n} x_i, 0 < \ell_i \leq x_i \leq u_i, i = 1, 2 \ldots, n\}$ are nonconvex sets.

Assuming M_n is contained in the first orthant, trivial bounds for x_{n+1} are given by $\prod_{i=1}^{n} \ell_i$ and $\prod_{i=1}^{n} u_i$, respectively, and denoted by $\bar{\ell}_{n+1}$ and $\bar{u}_{n+1}$. In general, $\bar{\ell}_{n+1} \leq \ell_{n+1} < u_{n+1} \leq \bar{u}_{n+1}$; in the remainder, we use the notation $M_n^\star$ for the set M_n where $\ell_{n+1} = \bar{\ell}_{n+1}$ and $u_{n+1} = \bar{u}_{n+1}$.

We are interested in developing a convex superset C_n of M_n defined by a system of linear inequalities, therefore we seek a polyhedral set $C_n \supseteq M_n$. Our interest is motivated by the nonconvex Optimization problem

$$\mathbf{P} : \min\{cx : x \in X\},$$

where X is, in general, a nonconvex set. In order to solve problems like $\mathbf{P}$ to optimality, one needs to implicitly enumerate all local optima, for instance by branch-and-bound algorithms. The bounding algorithms in such approaches often relies on a linear relaxation of the nonconvex problem [**3, 4**], thus benefiting from tighter linear relaxations.

2. Linear inequalities for $n = 2$

Here we develop convex inequalities for the set $M_2 = \{(x_1, x_2, x_3) \in [l, u] : x_3 = x_1 x_2\}$. Although this is the simplest case, many considerations generalize easily to the case when $n > 2$.

2.1. Unbounded x_3

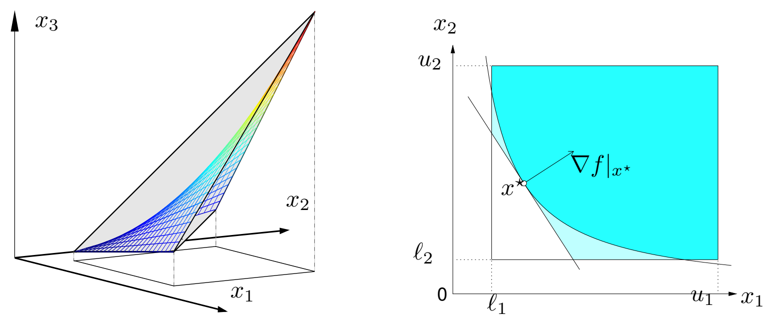
The following linear relaxation of $M_2^\star$ was introduced by McCormick [**2**] and shown to be its convex hull by Al-Khayyal and Falk [**1**]:

$$\begin{aligned}
x_3 &\geq \ell_2 x_1 + \ell_1 x_2 - \ell_1 \ell_2 \\
x_3 &\geq u_2 x_1 + u_1 x_2 - u_1 u_2 \\
x_3 &\leq \ell_2 x_1 + u_1 x_2 - \ell_1 u_2 \\
x_3 &\leq u_2 x_1 + \ell_1 x_2 - u_1 \ell_2,
\end{aligned}$$

and is depicted in Figure 1a for convenience. The shaded tetrahedron is the (polyhedral) convex hull, while $M_2^\star$ is shown in colors.

2.2. Nontrivial lower and upper bound for x_3

We consider first a finite lower bound, $\ell_3 > \bar{\ell}_3 = \ell_1 \ell_2$. The darker area in Figure 1b shows the projection of M_2 onto (x_1, x_2), that is, the set $P_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \ell_i \leq x_i \leq u_i, i = 1, 2, x_1 x_2 \geq \ell_3\}$. It is safe to assume here that $\ell_3 \leq \ell_1 u_2$ and $\ell_3 \leq u_1 \ell_2$, as otherwise a valid lower bound for x_1 (resp. x_2) would be $\ell_3/u_2 > \ell_1$ (resp $\ell_3/u_1 > \ell_2$), or equivalently, the upper left (resp. the lower right) corner of the



(a) Convex hull of $M_2^\star$.

(b) Projections of M_2 with non-trivial lower bound ℓ_3.

Figure 1

bounding box in Figure 1b would be cut out by the convex set $x_1x_2 \geq \ell_3$. Similarly we assume that $u_3 \geq \ell_1 u_2$ and $u_3 \geq u_1\ell_2$.

Projecting M_2 onto (x_1, x_2) gives a convex set P_2. Consider a point $x^\star$ on the curve $x_1x_2 = \ell_3$ with $\ell_1 \leq x_1^\star \leq u_1$ and $\ell_2 \leq x_2^\star = \ell_3/x_1^\star \leq u_2$. The tangent to the curve $x_1x_2 = \ell_3$ at $x^\star$ gives a linear inequality $a_1(x_1 - x_1^\star) + a_2(x_2 - x_2^\star) \geq 0$, that is valid for P_2. The coefficients a_1 and a_2 are given by the gradient of the function $\xi_2(x) = x_1x_2$ at $x^\star$, i.e., $a_1 = \frac{\partial\pi}{\partial x_1}|_{x^\star} = x_2^\star$ and $a_2 = \frac{\partial\pi}{\partial x_2}|_{x^\star} = x_1^\star$. The lighter area within the bounding box above the tangent line is the set of points satisfying the above linear constraint, which we rewrite here:

$$x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) \geq 0. \tag{3.1}$$

As this inequality is valid within P_2 and is independent from x_3, it is also valid for M_2. We can lift it as follows: the inequality

$$x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) + a(x_3 - \ell_3) \geq 0 \tag{3.2}$$

is clearly valid for $x_3 = \ell_3$. For it to be valid for M_2,

$$g(a_3) = \min\{x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) + a(x_3 - \ell_3) : (x_1, x_2, x_3) \in M_2\} \geq 0$$

must hold. It is easy to show that $g(a_3) = 0$ if $a \geq 0$ (a global optimum is given by $(x_1^\star, x_2^\star)$), hence $a < 0$ in all lifted inequalities (3.2).

We next show how to calculate a. First we will show how to get this coefficient and then we will prove that the inequalities we get are not dominated by any other valid inequality. To find the coefficient a in the inequality (3.2), intuitively we know that the plane

$$x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) + a(x_3 - \ell_3) = 0 \tag{3.3}$$

should touch the curve $x_1x_2 = u_3$ at exactly one point and first we want to find this point. Let's call this point $(\bar{x}_1, \bar{x}_2)$. If for a moment we disregard the bounds on x_1 and x_2, the fact that the plane (3.3) touches the curve $(\bar{x}_1, \bar{x}_2)$ at exactly one point means that the plane would be tangent to the curve $x_1x_2 = u_3$ at that point. This means that the gradient of the curve at $(\bar{x}_1, \bar{x}_2)$ is parallel to the projection of the normal to the plane onto the (x_1, x_2) space. The gradient of the curve $x_1x_2 = u_3$ at

$(\bar{x}_1, \bar{x}_2)$ is $(\bar{x}_2, \bar{x}_1)$ and the projection of the normal to the plane onto the (x_1, x_2) space is $(x_2^\star, x_1^\star)$. As a result we will have:

$$\exists \alpha \in \mathbb{R} : \bar{x}_2 = \alpha x_2^\star, \bar{x}_1 = \alpha x_1^\star. \tag{3.4}$$

But we know that $\bar{x}_1\bar{x}_2 = u_3$, so we will have $\bar{x}_1\bar{x}_2 = u_3 = \alpha^2 x_1^\star x_2^\star = \alpha^2 \ell_3$, and as a result

$$\alpha = \sqrt{\frac{u_3}{\ell_3}}. \tag{3.5}$$

On the other hand we know that $(\bar{x}_1, \bar{x}_2)$ is on the plane (3.3), which implies

$$x_2^\star(\bar{x}_1 - x_1^\star) + x_1^\star(\bar{x}_2 - x_2^\star) + a(u_3 - \ell_3) = 0,$$

As a result,

$$x_2^\star(\alpha x_1^\star - x_1^\star) + x_1^\star(\alpha x_2^\star - x_2^\star) + a(u_3 - \ell_3) = 0. \tag{3.6}$$

By (3.5) and (3.6), the value for a is $\frac{2(1-\sqrt{\frac{u_3}{\ell_3}})\ell_3}{u_3-\ell_3}$.

Notice that the value of a does not depend on the value of $x_1^\star$ and $x_2^\star$ and is less than zero. Intuitively this value for a will give a tight valid inequality for M_2 if $\ell_1 \le \bar{x}_1 \le u_1$ and $\ell_2 \le \bar{x}_2 \le u_2$ (the proof will come later). But if $(\bar{x}_1, \bar{x}_2)$ is not in the domain of (x_1, x_2) we need to find a point on the curve $x_1x_2 = u_3$ at which the plane (3.3) touches the curve. Intuitively this point would be one of the end points of the curve $x_1x_2 = u_3; \ell_1 \le x_1 \le u_1, \ell_2 \le x_2 \le u_2$. In fact that point will be the one which is closer to $(\bar{x}_1, \bar{x}_2)$.

Again consider the curve $x_1x_2 = u_3$. On this curve we know that:

$$x_1 \ge \frac{u_3}{u_2} = \bar{l}_1; \quad x_2 \ge \frac{u_3}{u_1} = \bar{l}_2; \quad x_1 \le \frac{u_3}{\ell_2} = \bar{u}_1; \quad x_2 \le \frac{u_3}{\ell_1} = \bar{u}_2.$$

Then the real bounds over x_1 and x_2 on the curve $x_1x_2 = u_3$ would be $\ell_1^\star \le x_1 \le u_1^\star$, $\ell_1^\star = \max(\bar{l}_1, \ell_1)$, and $u_1^\star = \min(\bar{u}_1, u_1)$; $\ell_2^\star \le x_2 \le u_2^\star$, $\ell_2^\star = \max(\bar{l}_2, \ell_2)$, and $u_2^\star = \min(\bar{u}_2, u_2)$. However, based on the assumptions on the upper bound of x_3 (see Section 2.2)

$$\ell_1^\star = \frac{u_3}{u_2}; \quad u_1^\star = u_1; \quad \ell_2^\star = \frac{u_3}{u_1}; \quad u_2^\star = u_2. \tag{3.7}$$

As a result, the end points of the curve $x_1x_2 = u_3$ in M_2 are $(\frac{u_3}{u_2}, u_2)$ and $(u_1, \frac{u_3}{u_1})$. It's easy to see that if $\alpha x_1^\star \ge u_1$ and $\alpha x_2^\star \le \frac{u_3}{u_1}$, the the right end point would be $(u_1, \frac{u_3}{u_1})$, and if $\alpha x_1 \le \frac{u_3}{u_2}$ and $\alpha x_2 \ge u_2$, the right end point would be $(\frac{u_3}{u_2}, u_2)$. So if $\alpha x_1^\star \ge u_1$ and $\alpha x_2^\star \le \frac{u_3}{u_1}$ and $\frac{u_1}{\alpha} \le x_1^\star \le \frac{\ell_3}{\ell_2}$, the plane (3.3) would touch the curve $x_1x_2 = u_3$ at $(u_1, \frac{u_3}{u_1})$ and therefore we will have:

$$x_2^\star(u_1 - x_1^\star) + x_1^\star(\frac{u_3}{u_1} - x_2^\star) + a(u_3 - \ell_3) = 0, \tag{3.8}$$

and as a result

$$a = \frac{x_2^\star(u_1 - x_1^\star) + x_1^\star(\frac{u_3}{u_1} - x_2^\star)}{\ell_3 - u_3}. \tag{3.9}$$

On the other hand, if $\alpha x_1 \le \frac{u_3}{u_1}$ and $\alpha x_2 \ge u_2$ and $\ell_1 \le x_1 \le \frac{u_3}{\alpha u_2}$, the plane (3.3) would touch the curve $x_1x_2 = u_3$ at $(\frac{u_3}{u_2}, u_2)$ and we will have:

$$x_2^\star(\frac{u_3}{u_2} - x_1^\star) + x_1^\star(u_2 - x_2^\star) + a(u_3 - \ell_3) = 0, \tag{3.10}$$

hence,

$$a = \frac{x_2^\star(\frac{u_3}{u_2} - x_1^\star) + x_1^\star(u_2 - x_2^\star)}{\ell_3 - u_3}. \tag{3.11}$$

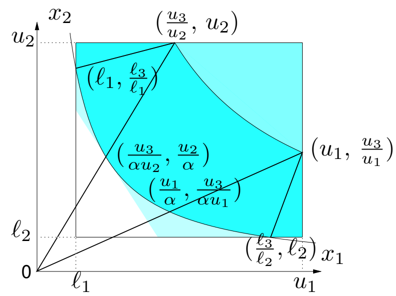


Figure 2

Now we need to deal with two other cases; $\frac{\ell_3}{\ell_2} \leq x_1^\star \leq u_1$, $x_2^\star = \ell_2$ and $x_1^\star = \ell_1$, $\frac{\ell_3}{\ell_1} \leq x_2^\star \leq u_2$. First consider the case which $\frac{\ell_3}{\ell_2} \leq x_1^\star \leq u_1$, $x_2^\star = \ell_2$. Intuitively we can see that the plane which goes through the points $(\frac{\ell_3}{\ell_2}, \ell_2, \ell_3)$, $(u_1, \ell_2, u_1\ell_2)$, and $(u_1, \frac{u_3}{u_1}, u_3)$ gives an inequality which is valid for M_2 and is not dominated by any other inequality. Similarly for the second case in which $x_1^\star = \ell_1$, $\frac{\ell_3}{\ell_1} \leq x_2^\star \leq u_2$, the plane which goes through the points $(\ell_1, \frac{\ell_3}{\ell_1}, \ell_3)$, $(\ell_1, u_2, \ell_1 u_2)$, and $(\frac{u_3}{u_2}, u_2, u_3)$ gives an inequality which is valid for M_2 and is not dominated by any other inequality.

In summary, as can be seen in Figure 2, if $\frac{u_3}{\alpha u_2} < x_1^\star < \frac{u_1}{\alpha}$ and $\frac{u_3}{\alpha u_1} < x_2^\star < \frac{u_2}{\alpha}$, then the point $(\bar{x}_1, \bar{x}_2)$ (the point at which the plane (3.3) touches the curve $x_1 x_2 = u_3$) will have $\frac{u_3}{u_2} < \bar{x}_1 < u_1$ and $\frac{u_3}{u_1} < \bar{x}_2 < u_2$. Otherwise, $(\bar{x}_1, \bar{x}_2)$ happens at either $(\frac{u_3}{u_2}, u_2)$ or $(u_1, \frac{u_3}{u_1})$.

References

1. F.A. Al-Khayyal and J.E. Falk. Jointly constrained biconvex programming. *Mathematics of Operations Research*, 8:273–286, 1983.
2. G.P. McCormick. *Nonlinear programming: theory, algorithms and applications.* John Wiley & sons, 1983.
3. E.M.B. Smith and C.C. Pantelides. Global optimisation of nonconvex MINLPs. *Computers & Chemical Engineering*, 21:S791–S796, 1997.
4. M. Tawarmalani and N.V. Sahinidis. Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Mathematical Programming*, 99(3):563–591, 2004.